FAMILIES INDEX THEOREM IN SUPERSYMMETRIC WZW MODEL AND TWISTED K-THEORY: THE $SU(2)$ CASE

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Abstract. The construction of twisted K-theory classes on a compact Lie group is reviewed using the supersymmetric Wess-Zumino-Witten model on a cylinder. The Quillen superconnection is introduced for a family of supercharges parametrized by a compact Lie group and the Chern character is explicitly computed in the case of $SU(2)$. For large euclidean time, the character form is localized on a D-brane.

0. Introduction

Gauge symmetry breaking in quantum field theory is described in terms of families index theory. The Atiyah-Singer index formula gives via the Chern character cohomology classes in the moduli space of gauge connections and of Riemann metrics. In particular, the 2-form part is interpreted as the curvature of the Dirac determinant line bundle, which gives an obstruction to gauge covariant quantization in the path integral formalism. The obstruction depends only on the K-theory class of the family of operators.

In the Hamiltonian quantization odd forms on the moduli space become relevant, [CMM]. The obstruction to gauge covariant quantization comes from the 3-form part of the character. The 3-form is known as the Dixmier-Douady class and is

also the (only) characteristic class of a gerbe; this is the higher analogue of the first Chern class (in path integral quantization) classifying complex line bundles.

The next step is to study families of "operators" which are only projectivly defined; that is, we have families of hamiltonians which are defined locally in the moduli space but which refuse to patch to a globally defined family of operators. The obstruction is given by the Dixmier-Douady class, an element of integral third cohomology of the moduli space. On the overlaps of open sets the operators are related by a conjugation by a projective unitary transformation. This leads to the definition of twisted K-theory, [DK], [Ro].

In the present paper we shall first review the basic definitions of both ordinary K-theory and twisted K-theory in Section 1. In Section 2 the construction of twisted (equivariant) K-theory classes on compact Lie groups is outlined using a supersymmetric model in $1+1$ dimensional quantum field theory. Finally, in Section 3 the Quillen superconnection formula is applied to the projective family of Fredholm operators giving a Chern character alternatively with values in Deligne cohomology on the base or in global twisted de Rham forms, [BCMMS]. The use of Quillen superconnection has been proposed in general context of twisted K-theory in [Fr], but here we will give the details in simple terms using the supersymmetric Wess–Zumino-Witten model.

The Quillen superconnection can be modified (Prop. 1) to give a map from twisted K-theory to twisted cohomology on the base space. However, the (nonequivariant) twisted K-theory of compact Lie groups is all torsion [Br, Do] and the twisted cohomology vanishes. Nevertheless it turns out that, at least in the case of the group $SU(2)$, the cocycle evaluated from the superconnection formula contains more information than its twisted cohomology class. We show that the de Rham form is integrally quantized as the dimension of the relevant $SU(2)$ representation times the basic 3-form on the group manifold. We also construct a map (not restricted to the case of $SU(2)$) from twisted K-theory to gerbes over the base, as defined in terms of local line bundles, modulo the twisting gerbe. In the case of $SU(2)$ this map agrees with the known identification $K^1(SU(2),k) = \mathbb{Z}/k\mathbb{Z}$, [Ro].

In the limit $\to \infty$ for the scaling parameter in the superconnection, the support

of the form is localized on a 'D-brane', a quantized conjugacy class on $SU(2)$, [GR].

1. Twist in K-theory by a gerbe class

Let M be a compact manifold and P a principal bundle over M with structure group $PU(H)$, the projective unitary group of a complex Hilbert space H. We shall consider the case when H is infinite dimensional. The characteristic class of P is represented by an element $\Omega \in H^3(M, \mathbb{Z})$, the Dixmier-Douady class.

Choose a open cover $\{U_\alpha\}$ of M with local transition functions $g_{\alpha\beta} : U_{\alpha\beta} \to PU(H)$ of the bundle P.

In the case of a good cover we can even choose lifts $\hat{g}_{\alpha\beta} : U_{\alpha\beta} \to U(H)$, to the unitary group in the Hilbert space H, on the overlaps $U_{\alpha\beta} = U_\alpha \cap U_\beta$, but then we only have

$$\hat{g}_{\alpha\beta}\hat{g}_{\beta\gamma}\hat{g}_{\gamma\alpha} = \sigma_{\alpha\beta\gamma} \cdot \mathbf{1}$$

for some $\sigma_{\alpha\beta\gamma} : U_{\alpha\beta\gamma} \to S^1$ where $U_{\alpha\beta\gamma} = U_\alpha \cap U_\beta \cap U_\gamma$ and $\mathbf{1}$ is the identity operator.

Complex K-theory classes on M may be viewed as homotopy classes of maps $M \to Fred$, to the space of Fredholm operators in an infinite-dimensional complex Hilbert space H. This defines what is known as $K^0(M)$. The other complex K-theory group is $K^1(M)$ and this is defined by replacing $Fred$ by $Fred_*$, the space of self-adjoint Fredholm operators with both positive and negative essential spectrum.

The twisted K-theory classes are here defined as homotopy classes of sections of a fiber bundle $\mathcal{Q}$ over M with model fiber equal to either $Fred$ or $Fred_*$. One sets

$$\mathcal{Q} = P \times_{PU(H)} Fred,$$

and similarly for $Fred_*$, where the $PU(H)$ action on $Fred$ is simply the conjugation by a unitary transformation $\hat{g}$ corresponding to $g \in PU(H)$.

We denote by $K^*(M, [\Omega])$ the twisted K theory classes, the twist given by P.

Using local trivializations a section is given by a family of maps $\psi_\alpha : U_\alpha \to Fred$ such that

$$\psi_\beta(x) = \hat{g}_{\alpha\beta}^{-1}(x)\psi_a(x)\hat{g}_{\alpha\beta}(x)$$

on the overlaps $U_{\alpha\beta}$.

2. Supersymmetric construction of $K(M,[\Omega])$

We recall from [Mi1] the construction the operator Q_A as a sum of a 'free' supercharge Q and an interaction term $\hat{A}$ in (2.7) acting in H. The Hilbert space H is a tensor product of a 'fermionic' Fock space H_f and a 'bosonic' Hilbert space H_b. Let G be a connected, simply connected simple compact Lie group of dimension N and $\mathbf{g}$ its Lie algebra. The space H_b carries an irreducible representation of the loop algebra $L\mathbf{g}$ of level k where the highest weight representations of level k are classified by a finite set of G representations (the basis of Verlinde algebra) on the 'vacuum sector'.

In a Fourier basis the generators of the loop algebra are T_n^a where $n \in \mathbb{Z}$ and $a = 1, ..., \dim G = N$. The commutation relations are

$$[T_n^a, T_m^b] = \lambda_{abc} T_{n+m}^c + \frac{k}{4} n \delta_{ab} \delta_{n,-m}, \tag{2.1}$$

where the λ_{abc}'s are the structure constant of $\mathbf{g}$; in the case when $\mathbf{g}$ is the Lie algebra of $SU(2)$ the nonzero structure constants are completely antisymmetric and we use the normalization $\lambda_{123} = \frac{1}{\sqrt{2}}$, corresponding to an orthonormal basis with respect to -1 times the Killing form. This means that in this basis the Casimir invariant $C_2 = \sum_{a,b,c} \lambda_{abc}\lambda_{acb}$ takes the value $-N$.

In a unitary representation of the loop group we have the hermiticity relations

$$(T_n^a)^* = -T_{-n}^a$$

With this normalization of the basis, for $G = SU(2)$, k is a nonnegative integer and $2j = 0, 1, 2, ..., k$ labels the possible irreducible representations of $SU(2)$ on the vacuum sector. The case $k = 0$ corresponds to a trivial representation and we shall assume in the following that k is strictly positive. In general the level k is quantized as an integer x times twice the square of the length of the longest root with respect to the dual Killing form (this unit is in our normalization equal to 1 in the case

$G = SU(2)$); alternatively, we can write $k = 2x/h^\wedge$, where $h^\wedge$ is the dual Coxeter number of the Lie algebra $\mathbf{g}$.

The space H_f carries an irreducible representations of the canonical anticommutation relations (CAR),

$$\psi_n^a \psi_m^b + \psi_m^b \psi_n^a = 2\delta_{ab}\delta_{n,-m}, \tag{2.2}$$

and $(\psi_n^a)^* = \psi_{-n}^a$. The representation is fixed by the requirement that there is an irreducible representation of the Clifford algebra $\{\psi_0^a\}$ in a subspace $H_{f,vac}$ such that $\psi_n^a v = 0$ for $n < 0$ and $v \in H_{f,vac}$.

The central extension of the loop algebra at level 2 is represented in H_f through the operators

$$K_n^a := -\frac{1}{4} \sum_{b,c;m\in\mathbb{Z}} \lambda_{abc} \psi_{n-m}^b \psi_m^c, \tag{2.3}$$

which satisfy

$$[K_n^a, K_m^b] = \lambda_{abc} K_{n+m}^c + \frac{1}{2} n \delta_{ab} \delta_{n,-m}. \tag{2.4}$$

We set $S_n^a = \mathbf{1} \otimes T_n^a + K_n^a \otimes \mathbf{1}$. This gives a representation of the loop algebra; in the case $G = SU(2)$ the level is $k+2$ in the tensor product $H = H_f \otimes H_b$. In the parametrization of the level by the integer x this means that we have a level shift $x \mapsto x' = x + h^\wedge$.

Next we define the supercharge operator

$$Q := i \sum_{a,n} \left(\psi_n^a T_{-n}^a + \frac{1}{3} \psi_n^a K_{-n}^a \right). \tag{2.5}$$

This operator satisfies $Q^2 = h$, where h is the hamiltonian of the supersymmetric Wess-Zumino-Witten model,

$$h := \underbrace{-\sum_{a,n} :T_n^a T_{-n}^a:}_{=:h_b} + \frac{k+2}{2} \cdot \underbrace{\frac{1}{4} \sum_{a,n} :n\psi_n^a \psi_{-n}^a:}_{=:h_f} + \frac{N}{24} = h_b + 2\tilde{k} h_f + \frac{N}{24}, \tag{2.6}$$

where $\tilde{k} := \frac{k+2}{4}$ and the normal ordering :: means that the operators with negative Fourier index are placed to the right of the operators with positive index, $:\psi_{-n}^a \psi_n^b:$

$= -\psi^b_n \psi^a_{-n}$ if $n > 0$ and $: AB := AB$ otherwise. In the case of the bosonic currents T^a_n the sign is $+$ on the right-hand-side of the equation.

Finally, the gauged supercharge operator Q_A is defined as

$$Q_A := Q + i\tilde{k} \sum_{a,n} \psi^a_n A^a_{-n} \tag{2.7}$$

where the A^a_n's are the Fourier components of the $\mathbf{g}$-valued function A in the basis T^a_n. We denote the mapping $A \mapsto Q_A$ by $Q_\bullet$.

The basic property of the family of self-adjoint Fredholm operators Q_A is that it is equivariant with respect to the action of the central extension of the loop group LG. Any element $w \in LG$ is represented by a unitary operator $S(w)$ in H but the phase of $S(w)$ is not uniquely determined. The equivariance property is

$$S(w^{-1}) Q_A S(w) = Q_{A^w} \tag{2.8}$$

with $A^w = w^{-1} A w + w^{-1} dw$. For the Lie algebra we have the relations

$$[S^a_n, Q_A] = i\tilde{k}\Big(n\psi^a_n + \sum_{b,c;m} \lambda_{abc} \psi^b_m A^c_{n-m} \Big) \tag{2.9}$$

The group LG can be viewed as a subgroup of the group $PU(H)$ through the projective representation S and occasionally we write w instead of $S(w)$. [In order that the embedding is continuous in operator norm in the Hilbert space of a positive energy representation, one should replace LG by the Sobolev completion with respect to the weight $1/2$, [PS].] The space $\mathcal{A}$ of smooth vector potentials on the circle is the total space for a principal bundle with fiber $\Omega G \subset LG$, the group of based loops at 1. Since now $\Omega G \subset PU(H)$, $\mathcal{A}$ may be viewed as a reduction of a $PU(H)$ principal bundle over G. The ΩG action by conjugation on the Fredholm operators in H defines an associated fiber bundle $\mathcal{Q}$ over G and the family of operators Q_A defines a section of this bundle. Thus $\{Q_A\}$ is a twisted K-theory class over G where the twist is determined by the level $k+2$ projective representation of LG.

Actually, there is additional gauge symmetry due to constant global gauge transformations. For this reason the construction above leads to elements in the G-equivariant twisted K-theory $K^*_G(G, [\Omega])$, where the G-action on G is the conjugation by group elements. The class $[\Omega]$ is represented by the form $\frac{(k+2)}{24\pi^2}\mathrm{tr}\,(g^{-1}dg)^3$ where the trace is computed in the defining representation of $SU(2)$. It happens that in the case of $SU(2)$ the construction gives all generators for both equivariant and nonequivariant twisted K-theories, but not for other compact Lie groups.

3. Quillen superconnection

Let Q_A be the supercharge associated to the vector potential A on the circle, with values in the Lie algebra $\mathbf{g}$. Recall that this transforms as

$$\hat{g}^{-1}Q_A\hat{g} = Q_{A^g}$$

with respect to $g \in LG \subset PU(H)$. Consider the trivial Hilbert bundle over $\mathcal{A}$ with fiber H, the operators Q_A acting in the fibers. Define a covariant differentiation ∇ acting on the sections of the bundle, $\nabla := \delta + \hat{\omega}$ where δ is the exterior differentiation on $\mathcal{A}$ and $\hat{\omega}$ is a connection 1-form defined as follows. First, any vector potential on the circle can be uniquely written as $A = f^{-1}df$ for some smooth function $f : [0,1] \to G$ such that $f(0) = 1$; here we parametrize S^1 with parameter $y \in [0,1]$ such that any element of S^1 is of the form $e^{2\pi iy}$. A tangent vector δf at f is then represented by a function $v : [0,1] \to \mathbf{g}$ such that $v(0) = 0$ with periodic derivatives at the end points and $v = f^{-1}\delta f$. We set, [CM],

$$\omega_f(\delta f) := f^{-1}\delta f - \alpha f^{-1}(\delta f(1)f(1)^{-1})f \tag{3.1}$$

where α is a fixed smooth real valued function on $[0,1]$ such that $\alpha(0) = 0, \alpha(1) = 1$ and all derivatives equal to zero at the end points. The point of the second term in (3.1) is that it makes the whole expression periodic so that ω takes values in $L\mathbf{g}$. Then $\hat{\omega}_f(\delta f)$ is defined by the projective representation S of $L\mathbf{g}$ in H.

The gauge transformation $A \mapsto A^g$ corresponds to the right translation $r_g(f) = fg$, which sends $\omega_f(\delta f)$ to $g^{-1}\omega_f(\delta f)g$. However, for the quantized operator $\hat{\omega}$ we

get an additional term. This is because of the central extension $\widehat{LG}$ which acts on $\hat{\omega}$ through the adjoint representation. One has

$$\hat{g}^{-1}\hat{\omega}\hat{g} = \widehat{g^{-1}\omega g} + \gamma(\omega, g)$$

with

$$\gamma(\omega, g) := \frac{k+2}{8\pi}\int_{S^1} \langle \omega, dgg^{-1}\rangle_K.$$

The bracket $\langle \cdot\,, \cdot\rangle_K$ is the Killing form on $\mathbf{g}$. But one checks that the modified 1-form

$$\hat{\omega}_c|_f := \hat{\omega}_f - \frac{k+2}{8\pi}\int_{S^1} \langle \omega_f, f^{-1}df\rangle_K$$

transforms in a linear manner,

$$\hat{g}^{-1}\hat{\omega}_c\hat{g} = \widehat{r_g\omega_c}. \tag{3.2}$$

Here r_g denotes the right action of ΩG on $\mathcal{A}$ and the induced right action on connection forms. We would like to construct characteristic classes on the quotient space $\mathcal{A}/\Omega G$ from classes on $\mathcal{A}$ using the equivariance property (3.2). First, we can construct a Quillen superconnecton [Qu] as the form of mixed degree

$$D_t := \sqrt{t}Q_\bullet + \delta + \hat{\omega}_c - \frac{1}{4\sqrt{t}}\langle\psi, F\rangle, \tag{3.3}$$

where F is the Lie algebra valued curvature form computed from the connection ω and $\langle\psi, F\rangle := \sum \psi_n^a F_{-n}^a$ where F_{-n}^a's are the Fourier coefficients of F. Formally, this expression is the same as the Bismut superconnection for families of Dirac operators, [Bi]. Here t is a free positive real scaling parameter. This is introduced since in the case of Bismut superconnection one obtains the Atiyah-Singer families index forms in the limit $t \to 0$ from the formula (3.4) or (3.5) below.

When $\dim G$ is even, we define a family of closed differential forms on $\mathcal{A}$ from

$$\Theta^t := \mathrm{tr}_s\, e^{-(\sqrt{t}Q_\bullet + \delta + \hat{\omega}_c - \frac{1}{4\sqrt{t}}\langle\psi, F\rangle)^2}. \tag{3.4}$$

In this case, the supertrace is defined as $\mathrm{tr}_s(\cdot) = \mathrm{tr}\,\Gamma(\cdot)$. Here Γ is the grading operator with eigenvalues ± 1. It is defined uniquely up to a phase ± 1 by the requirement that it anticommutes with each ψ_n^a and commutes with the algebra T_n^a. To get integral forms the n-form part of Θ^t should be multiplied by $(1/2\pi i)^{n/2}$.

In the odd case the above formula has to be modified:

$$\Theta^t := \mathrm{tr}^\nu\, e^{-(\nu\sqrt{t}Q_\bullet+\delta+\hat\omega_c-\frac{\nu}{4\sqrt{t}}\langle\psi,F\rangle)^2}, \tag{3.5}$$

where ν is an odd element, $\nu^2 = 1$, anticommuting with odd differential forms and commuting with Q_A, and the trace tr^ν extracts the operator trace of the coefficients of the linear term in ν. In this case the n-form part should be multiplied by $\sqrt{2i}(1/2\pi i)^{n/2}$.

The problem with the expressions (3.4) and (3.5) is that they cannot pushed down to the base $G = \mathcal{A}/\Omega G$. The obstruction comes from the transformation property

$$\begin{aligned}&\sqrt{t}Q_{A^g} + \delta + \widehat{r_g\omega_c} - \frac{1}{4\sqrt{t}}\langle\psi, r_g F\rangle\\ &= \hat g^{-1}(\sqrt{t}Q_A + \delta + \hat\omega_c - \frac{1}{4\sqrt{t}}\langle\psi, F\rangle)\hat g + \hat g^*\theta,\end{aligned} \tag{3.6}$$

where θ is the connection 1-form on $\widehat{LG}$ corresponding to the curvature form c on LG, defined by the central extension. Here $\hat g$ is a local $\widehat{LG}$ valued function on the base G, implementing a change of local section $G \to \mathcal{A}$. This additional term is the difference

$$\hat g^*\theta = \widehat{g^{-1}\delta g} - \hat g^{-1}\delta\hat g,$$

where the first term on the right comes from the transformation of the connection form $\hat\omega_c$ with respect to a local gauge transformation g. Taking the square of the transformation rule (3.6) we get

$$(r_g D_t)^2 = \hat g^{-1} D_t^2 \hat g + g^* c. \tag{3.7}$$

The last term on the right arises as

$$\delta\hat g^*\theta + (\hat g^*\theta)^2 = \hat g^*\delta\theta = \hat g^* c = g^* c,$$

where in the last step we have used the fact that the curvature of a circle bundle is a globally defined 2-form c on the base, and thus does not depend on the choice of the lift $\hat g$ to $\widehat{LG}$.

Proposition 1. *Let U_α and U_β be two open sets in G with local sections ψ_α, ψ_β to the total space $\mathcal{A}$ of the ΩG principal bundle $\mathcal{A} \to G$. Let $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \Omega G$ be the local gauge transformation tranforming ψ_α to ψ_β. Then the pull-back forms Θ^t_α and Θ^t_β are related on $U_\alpha \cap U_\beta$ as*

$$\Theta^t_\beta = \psi^*_\beta \Theta^t = e^{-g^*_{\alpha\beta} c} \Theta^t_\alpha.$$

Proof. Since the curvature is closed, $\delta c = 0$, the term $g^* c$ on the right in (3.7) commutes with the rest and therefore can be taken out as a factor $\exp(-g^* c)$ in the exponential of the square of the transformed superconnection.

Remark. It is an immediate consequence of the Proposition 1 that the 1-form part $\Theta^t[1]$ of Θ^t is a globally defined form on the base G. We can view this as the generalization of the differential of the families η invariant, governing the spectral flow along closed loops in the parameter space; in fact, in the classical case of Bismut-Freed superconnection for families of Dirac operators this is exactly what one gets from the Quillen superconnection formula. We can write $\pi^{-1/2}\Theta^t[1] = h^{-1} dh/2\pi i$ with $\log h = 2\pi i \eta$. Note that η is only continuous modulo integers. Thinking of η as the spectral asymmetry, we normalize it by setting $\eta(A) = 0$ for the vector potential $A = 0$, or on the base, for the trivial holonomy $g = 1$.

In the odd case we can relate the calculation of $\Theta^t[3]$ to the computation of the Deligne class in twisted K-theory, [Mi2].

On the overlap $U_{\alpha\beta}$ we have from the Proposition:

$$\Theta^t_\beta[3] = \Theta^t_\alpha[3] - g^*_{\alpha\beta} c \wedge \Theta^t_\alpha[1]. \tag{3.8}$$

This gives

$$\Theta^t_\alpha[3] - \Theta^t_\beta[3] = d(\hat{g}^*_{\alpha\beta}\theta \wedge \Theta^t[1]) \equiv d\omega^2_{\alpha\beta}. \tag{3.9}$$

Using $\hat{g}_{\alpha\beta}\hat{g}_{\beta\gamma}\hat{g}_{\gamma\alpha} = \sigma_{\alpha\beta\gamma}$ we get

$$\omega^2_{\alpha\beta} - \omega^2_{\alpha\gamma} + \omega^2_{\beta\gamma} = (\sigma^{-1}_{\alpha\beta\gamma} d\sigma_{\alpha\beta\gamma}) \wedge \Theta^t[1]. \tag{3.10}$$

Choose a function $h : G \to S^1$ as in the Remark, $\pi^{-1/2}\Theta^t[1] = h^{-1} dh/2\pi i$.

Next the Čech coboundary of the cochain $\{\omega^2_{\alpha\beta}\}$ in (3.10) can be written as

$$d\omega^1_{\alpha\beta\gamma} = d(\log(\sigma_{\alpha\beta\gamma})h^{-1}dh). \tag{3.11}$$

Defining $a_{\alpha\beta\gamma\delta} = \log(\sigma_{\beta\gamma\delta}) - \log(\sigma_{\alpha\gamma\delta}) + \log(\sigma_{\alpha\beta\delta}) - \log(\sigma_{\alpha\beta\gamma})$ we can write

$$(\partial\omega^1)_{\alpha\beta\gamma\delta} = h^{-a_{\alpha\beta\gamma\delta}} dh^{a_{\alpha\beta\gamma\delta}} \tag{3.12}$$

where ∂ denotes the Čech coboundary operator. Thus the collection $\{\Theta^t_\alpha[3], \omega^2_{\alpha\beta}, \omega^1_{\alpha\beta\gamma}, h^{a_{\alpha\beta\gamma\delta}}\}$ defines a Deligne cocycle on the manifold G with respect to the given open covering $\{U_\alpha\}$.

The system of closed local forms obtained from the Chern character formula (3.5) can be modified to a system of *global forms* by multiplication

$$\tilde{\Theta}^t := e^{-\theta_\alpha} \wedge \Theta^t, \tag{3.13}$$

where θ_α is the 2-form potential, $d\theta_\alpha = \Omega$ on U_α. One checks easily that now

$$(d + \Omega)\tilde{\Theta}^t = 0. \tag{3.14}$$

Although the operator $d + \Omega$ squares to zero and can thus be used to define a cohomology theory, [BCMMS], the Chern character should not be viewed to give a map to this twisted cohomology theory. In fact, the twisted cohomology over complex numbers vanishes for simple compact Lie groups. For this reason, in order to hope to get nontrivial information from the Chern character, one should look for a refinement of the twisted cohomology. In fact, there is another integral version of twisted cohomology proposed in [At]. In that version one studies the ordinary integral cohomology modulo the ideal generated by the Dixmier-Douady class Ω. At least in the case of $SU(2)$ it is an experimental fact that the twisted (nonequivariant) K-theory as an abelian group is isomorphic to the twisted cohomology in this latter sense. There is a similar result for other compact Lie groups, but the 3-cohomology class used to define the twisting in cohomology is in general not the original twisting gerbe class; both are integral multiples of a basic 3-form, but the coefficients differ, except for the case of $SU(2)$, [Br, Do].

One can explicitly see why the integral cohomology mod Ω is relevant for twisted K-theory by the following construction in the odd case. First we replace the space $Fred_*$ by the homotopy equivalent space $\mathcal{U}_1$ consisting of $\mathbf{1}+$trace-class unitaries on H. An ordinary K-theory class on X is a homotopy class of maps $m : X \to \mathcal{U}_1$. In this representation the Chern character defines a sequence of cohomology classes on X by pulling back the generators $\mathrm{tr}\,(g^{-1}dg)^{2n+1}$ of the cohomology of $\mathcal{U}_1$. In the twisted case we have only maps m_α on open sets U_α which are related by $m_\beta = g_{\alpha\beta}^{-1} m_\alpha g_{\alpha\beta}$ on overlaps.

In the case of $G = SU(2) = S^3$ we need only two open sets $U_\pm$, the slightly extended upper and lower hemispheres, and a map g_{-+} on the overlap $U_{-+} = U_- \cap U_+$ to the group $PU(H)$, of degree k. If now $m_- \equiv \mathbf{1}$ identically and the set of points x with $m_+(x) \neq \mathbf{1}$ (the support of m_+) is concentrated around the North pole, then the pair $m_\pm$ is related by the conjugation by $g = g_{-+}$ at the equator and at the same time it defines an ordinary K-theory class since the functions patch to a globally defined function on S^3. Let us assume that the winding number of $m : S^3 \to \mathcal{U}_1$ is k. Next form a continuous path $m_\pm(t)$ of representatives of twisted K-theory classes starting from $m(1) = m$ and ending at the trivial class represented by the constant function $m(0) = \mathbf{1}$. Let ρ be a smooth function on S^3 which is equal to 1 on the overlap U_{-+} and zero in small open neighborhoods $V_\pm$ of the poles. We can also extend the domain of definition of $g(x)$ to a larger set $U_+ \setminus V_+$. For $0 \leq t \leq 1$ define

$$[m_-(t)](x) := e^{2\pi i t\rho(x) P_0} \text{ and } [m_+(t)](x) := e^{2\pi i t \rho(x) P(x)}$$

where $P(x) = g^{-1}(x) P_0 g(x)$, with P_0 is a fixed rank one projection, $x \in U_{-+}$. These are smooth functions on $U_\pm$ respectively and are related by the conjugation by g on the overlap. But for $t = 0$ both are equal to the identity $\mathbf{1} \in \mathcal{U}_1$. On the other hand, at $t = 1$ the integral

$$\frac{1}{24\pi^2} \int_{S^3} \mathrm{tr}\,(m^{-1}dm)^3$$

is easily computed to give the value k. This paradox is explained by the fact that for the intermediate values $0 < t < 1$ the functions $m_\pm$ do not patch up to a global

function on S^3. Thus we have a homotopy joining a pair of (trivial) twisted K-theory classes corresponding to the pair of third cohomology classes $0, \Omega$ computed from the Chern character. This confirms the claim, at least in the case of $X = S^3$, that the values of the Chern character should be projected to the quotient $H^*(X, \mathbb{Z})/\mathbb{Z}\Omega$.

4. Quillen superconnection in the case of $SU(2)$

In this section, we consider the case $G = SU(2)$ and calculate a pull-back of $\Theta^t = \mathrm{tr}^\sigma e^{-D_t^2}$, with respect to a local section over $U_+ = SU(2) \setminus \{-1\}$, in the limit $t \to \infty$.

The basic idea is that in this case one can naturally associate to the twisted K-theory class a K-theory class, defined modulo the twisting gerbe. This method was used in [Mi2] to calculate a characteristic class for twisted K theory on 3-manifolds. Actually, on the level of 3-cohomology that method can be generalized to arbitrary simply connected compact Lie groups as follows.

Let $\{U_\alpha\}$ be an open cover of G which trivializes the bundle $\mathcal{A} \to G$. Let $\psi_\alpha : U_\alpha \to \mathcal{A}$ be local sections. We may take $\{U_\alpha\}$ as the open set of holonomies such that the real number α is not in the spectrum of Q_A (for any A in the fibers over U_α). Let $L'_{\alpha\beta}(A)$ be be the top exterior power of the finite dimensional spectral subspace $E_{\alpha\beta} : \alpha < Q_A < \beta$ and denote

$$L_{\alpha\beta} = \psi_\alpha^* L'_{\alpha\beta} \otimes (h_{\alpha\beta}^* c)^{-n_\beta}$$

where c is the level $\tilde{k}$ central extension cocycle of the loop group and $h_{\alpha\beta}$ is the transition function, $\psi_\beta = \psi_\alpha h_{\alpha\beta}$. The integers n_β will be determined below. We require that

$$L_{\alpha\beta} \otimes L_{\beta\gamma} \otimes L_{\gamma\alpha} = \mathbb{C},$$

canonically isomorphic to the trivial line bundle. First note that $\psi_\beta^* L'_{\beta\gamma} = \psi_\alpha^* L'_{\beta\gamma} \otimes (h_{\alpha\beta}^* c)^{n_{\beta\gamma}}$ by the projective action of the loop group on the spectral subspaces, where $n_{\beta\gamma} = \dim\, E_{\beta\gamma}$ for $\beta < \gamma$ and $n_{\gamma\beta} = -n_{\beta\gamma}$. Therefore

$$\begin{aligned} L_{\alpha\beta} \otimes L_{\beta\gamma} \otimes L_{\gamma\alpha} &= \psi_\alpha^* L'_{\alpha\beta} \otimes \left(\psi_\alpha^* L'_{\beta\gamma} \otimes (h_{\alpha\beta}^* c)^{n_{\beta\gamma}}\right) \\ &\quad \otimes \left(\psi_\alpha^* L'_{\gamma\alpha} \otimes (h_{\alpha\gamma}^* c)^{n_{\gamma\alpha}}\right) \otimes (h_{\alpha\beta}^* c)^{-n_\beta} \otimes (h_{\beta\gamma}^* c)^{-n_\gamma} \otimes (h_{\gamma\alpha}^* c)^{-n_\alpha} \\ &= \psi_\alpha^* (L'_{\alpha\beta} \otimes L'_{\beta\gamma} \otimes L'_{\gamma\alpha}) \otimes (h_{\alpha\beta}^* c)^{n_{\beta\gamma} - n_\beta} \otimes (h_{\beta\gamma}^* c)^{-n_\gamma} \otimes (h_{\gamma\alpha}^* c)^{-n_{\gamma\alpha} - n_\alpha}. \end{aligned}$$

The line bundles $L'_{\alpha\beta}$ form a cocycle as follows from $E_{\alpha\gamma} = E_{\alpha\beta} \oplus E_{\beta\gamma}$ and therefore the nontrivial part is a product of pull-backs of c. Using the group property of the central extension of the loop group we have $h^*_{\alpha\gamma}c = h^*_{\alpha\beta}c \otimes h^*_{\beta\gamma}c$ and collecting the factors the tensor product above becomes

$$(h^*_{\alpha\beta}c)^{n_{\beta\alpha}-n_\beta+n_\alpha} \otimes (h^*_{\beta\gamma}c)^{n_{\gamma\alpha}-n_\gamma+n_\alpha},$$

where we have taken into account $n_{\beta\gamma} + n_{\gamma\alpha} = n_{\beta\alpha}$ in the first factor, and this becomes trivial provided we can choose the locally constant functions n_α such that $n_{\alpha\beta} = n_\alpha - n_\beta$ on triple intersections. But since we assumed that the base G is simply connected, $H^1(G) = 0$ and the solution exists.

The solution is not uniquely defined: it is defined modulo adding to each n_α a constant. This corresponds to modifying the gerbe defined by the line bundles $L_{\alpha\beta}$ by a power of the gerbe defined by the local line bundles $h^*_{\alpha\beta}c$. But these latter line bundles correspond to the Dixmier-Douady class Ω. Thus we get a map from twisted K theory $K^1(G,\Omega)$ to $H^3(G,\mathbb{Z})/\mathbb{Z}\Omega$. This map needs be neither injective nor surjective. However, in the case of $G = SU(2)$ these groups are known to be equal.

We now explain why the Quillen superconnection formalism has to give the same result. The calculation of the characteristic class in [Mi2] was based on the following observation. As a classifying space for self-adjoint Fredholm operators, with both positive and negative essential spectrum, one can also use the Lie group G_1 consisting of unitary operators which differ from the unit by a trace class operator. The 3-cohomology of G_1 is generated by $\mathrm{tr}\,(g^{-1}dg)^3$. In general, the K^1 class of a map $\phi : X \to G_1$ is mapped to cohomology as the odd Chern character given by the de Rham forms $\phi^*\mathrm{tr}\,(g^{-1}dg)^{2i+1}$. In the twisted case we do not have a map from X to G_1 but a family of local maps which are related on intersections of open sets by conjugation by $PU(H)$ valued transition functions. But in the case $X = SU(2)$ one can deform the twisted map such that it becomes a globally well-defined map $SU(2) \to G_1$ and one gets a characteristic class $\phi^*\mathrm{tr}\,(g^{-1}dg)^3$ in twisted K theory. However, as explained in the end of previous Section, this class is defined only modulo the Dixmier-Douady class of the twisting bundle.

Going back to the $G = SU(2)$ case, in [Mi2] the map from twisted K-theory to untwisted K theory was explicitly performed by using the fact that the ΩG bundle over G is trivialized over a pair of open sets, $S^3_+ = S^3 \setminus \{N\}$ and D^3_N where S, N are the 'South' and 'North' poles on the 3-sphere and D^3_N is a small disk around the North pole. The transition functions are defined on a thickened 2-sphere close to N. After a deformation, the map $g_+ : S^3_+ \to G_1$ becomes constant equal to the unit element in G_1 and thus defines a global K^1 class on S^3 by pull-back from G_1. Applying the Quillen superconnection to this class (using instead of G_1 the space $Fred_*$ as a classifying space) must give the same cohomology class. However, we can do better than that:

Using the global $SU(2)$ gauge invariance of the construction of Fredholm operators, we know that the differential forms obtained from the Quillen superconnection must be invariant under conjugation by $SU(2)$. The cohomology class does not depend on the scaling parameter t. On the other hand, in the limit $t \to \infty$ the forms are supported only on the subset of parameters A such that the operator Q_A has zero eigenvalue. The zeros of Q_A are easily computed, [FHT2]. The elements of a simple compact Lie group G can be parametrized as $g = \exp(2\pi a)$, where a is in the Lie algebra of G. The constant vector potential $A \equiv a$ on the circle S^1 has the holonomy g. Since the construction of the operators Q_A is equivariant with respect to (constant) gauge transformations we may assume that a is in the Cartan subalgebra $\mathbf{h}$ of G. Actually, performing an additional gauge transformation one can require that

$$(a + \overset{\vee}{d}, \alpha_i) \leq 0$$

where α_i's are the simple roots of the affine Lie algebra based on G, $\vee$ means the duality transformation $\mathbf{h} \to \mathbf{h}^*$ defined by the Killing form, and $(\cdot, \cdot)$ is the inner product in $\mathbf{h}^*$ coming from the Killing form. Here $d = -i\frac{d}{d\theta}$ is the derivation of the loop algebra. Then the only zeros of Q_A are in the weight subspace $\lambda + \rho$ where λ is the highest weight of the G representation on the bosonic vacuum sector, ρ is half the sum of the positive roots (which is also a highest weight on the fermionic vacuum sector). The value of a corresponding to the nonzero kernel of Q_A is given

by

$$\tilde{k}\overset{\vee}{a} = -\lambda - \rho.$$

In particular, for $G = SU(2)$ we obtain $\overset{\vee}{a} = -2j/\tilde{k}$ where $2j = 0, 1, 2, \ldots$ and in our normalization $\tilde{k} = (k+2)/4$.

Let $\varphi \in [0, 2\pi[$ and $\mathbf{n} \in S^2$. Define a local section $\psi : SU(2) \setminus \{N\} \to \mathcal{A}$ by $\psi(\varphi, \mathbf{n}) = -\frac{i}{2}\varphi\mathbf{n} \cdot \sigma$ where $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ and σ_a's are the Pauli matrices. At the boundary $\varphi = 2\pi$ all the constant vector potentials $\psi(2\pi, \mathbf{n})$ have holonomy -1 around the circle.

At $\varphi = 2\pi$ the operators $Q_{\psi(\varphi,\mathbf{n})}$ are invertible and for this reason at the limit $t \to \infty$ the exponential of the Quillen superconnection vanishes: Thus in this limit the forms obtained from (3.5) become globally well-defined on $SU(2)$ despite the fact that $\psi : SU(2) \to \mathcal{A}$ is discontinuous at $\varphi = 2\pi$.

To conclude, since the 3-form part of Θ^t invariant under conjugation by $SU(2)$ and, in the limit $t \to \infty$, $\psi^*\Theta^t[3]$ becomes concentrated on the 2-sphere, the form must be proportional to the area 2-form of the sphere. The proportionality constant is uniquely determined by the cohomology class. This justifies the result of Theorem.

Let K be a compact (Hausdorff) space, and let $C(K)$ be a Banach space of continuous functions $u : K \to \mathbb{C}$ equipped with the sup norm. By the Riesz representation theorem, the topological dual $M(K)$ of $C(K)$ can be considered as a set of regular complex Borel measures on K. Equip $M(K)$ with the weak-star topology. In this topology, we say that a net $\{\mu_i\}_{i\in\mathcal{I}} \subseteq M(K)$, where $\mathcal{I}$ is a directed set, converges to a point $\mu \in M(K)$ if $\lim_{i\in\mathcal{I}} \int u d\mu_i = \int u d\mu$ for any $u \in C(K)$; we denote $\mu = \text{w}^*\text{-}\lim_{i\in\mathcal{I}} \mu_i$.

Let $\mathbf{area}(S^2) \in M(S^2)$ denote the measure defined by the area 2-form of S^2, $\delta_a \in M\big([0, 2\pi]\big)$ the Dirac measure concentrated on $a \in]0, 2\pi[$, and

$$\varphi_j^k := 2\pi\frac{2j+1}{4\tilde{k}} = 2\pi\frac{2j+1}{k+2} \in]0, 2\pi[.$$

Theorem. *When $s^*\Theta^t[3]$ is interpreted as a measure on $[0, 2\pi] \times S^2$*

$$\underset{t\to\infty}{\text{w}^*\text{-}\lim}\, \psi^*\Theta^t[3] = -\sqrt{\pi}i\left(j + \frac{1}{2}\right)\delta_{\varphi_j^k} \otimes \mathbf{area}(S^2).$$

To get an integral form, we multiply $\Theta^t[3]$ by $(\sqrt{\pi}i)^{-1}(2\pi)^{-1}$.

For completeness, next we represent the sketch of the direct but lengthy proof of Theorem taken from our previous (unpublished) version of the article, [MP].

Proof. Since

$$\begin{aligned} D_t^2 =& tQ_\bullet^2 + \sqrt{t}\nu \underbrace{\Big(-\delta Q_\bullet + [Q_\bullet, \hat{\omega}_c]\Big)}_{=:B_1\,\text{(1-form)}} + \underbrace{\Big(\delta\hat{\omega}_c + \hat{\omega}_c^2 - \frac{1}{4}\{Q_\bullet, \langle\psi, F\rangle\}\Big)}_{=:B_2\,\text{(2-form)}} \\ & - \frac{\nu}{4\sqrt{t}} \underbrace{\Big(-\delta\langle\psi, F\rangle + [\langle\psi, F\rangle, \hat{\omega}_c]\Big)}_{=:B_3\,\text{(3-form)}} = t\big(Q_\bullet^2 - E_t\big) \end{aligned} \tag{4.2}$$

where

$$E_t := -\frac{1}{t}\left(\sqrt{t}\nu B_1 + B_2 - \frac{\nu}{4\sqrt{t}} B_3\right), \tag{4.3}$$

using the perturbation series expansion

$$\mathrm{tr}^\nu e^{-t(Q_\bullet^2 - E_t)} = \mathrm{tr}^\nu e^{-tQ_\bullet^2} + \sum_{n=1}^{\infty} t^n \int_{\Delta_n} \mathrm{tr}^\nu \left(e^{-ts_1 Q_\bullet^2} E_t e^{-ts_2 Q_\bullet^2} E_t \cdots e^{-ts_{n+1} Q_\bullet^2}\right) ds_1 \cdots ds_n,$$

where Δ_n is the standard n-simplex, one gets

$$\begin{aligned} \Theta^t =& \mathrm{tr}^\nu e^{-tQ_\bullet^2} + t\int_{\Delta_1} \mathrm{tr}^\nu \left(e^{-ts_1 Q_\bullet^2} E_t e^{-ts_2 Q_\bullet^2}\right) ds_1 \\ & + t^2 \int_{\Delta_2} \mathrm{tr}^\nu \left(e^{-ts_1 Q_\bullet^2} E_t e^{-ts_2 Q_\bullet^2} E_t e^{-ts_3 Q_\bullet^2}\right) ds_1 ds_2 \\ & + t^3 \int_{\Delta_3} \mathrm{tr}^\nu \left(e^{-ts_1 Q_\bullet^2} E_t e^{-ts_2 Q_\bullet^2} E_t e^{-ts_3 Q_\bullet^2} E_t e^{-ts_4 Q_\bullet^2}\right) ds_1 ds_2 ds_3. \end{aligned}$$

The three form part of the above form is

$$\begin{aligned} \Theta^t[3] =& t\sqrt{t}\int_{\Delta_3} \mathrm{tr}\left(e^{-ts_1 Q_\bullet^2} B_1 e^{-ts_2 Q_\bullet^2} B_1 e^{-ts_3 Q_\bullet^2} B_1 e^{-ts_4 Q_\bullet^2}\right) ds_1 ds_2 ds_3 \\ & + \sqrt{t}\int_{\Delta_2} \mathrm{tr}\left(e^{-ts_1 Q_\bullet^2} B_1 e^{-ts_2 Q_\bullet^2} B_2 e^{-ts_3 Q_\bullet^2} + e^{-ts_1 Q_\bullet^2} B_2 e^{-ts_2 Q_\bullet^2} B_1 e^{-ts_3 Q_\bullet^2}\right) ds_1 ds_2 \\ & \frac{1}{4\sqrt{t}} \int_{\Delta_1} \mathrm{tr}\left(e^{-ts_1 Q_\bullet^2} B_3 e^{-ts_2 Q_\bullet^2}\right) ds_1. \end{aligned}$$

After long simplification (see, [MP]) one gets

$$\begin{aligned} \psi^*\Theta^t[3]_{\varphi,\mathbf{n}} =& -\frac{i\tilde{k}^2}{\pi^2\sqrt{2}} \sqrt{t} e^{-t[2\tilde{k}^2\tilde{\varphi}^2 - (2j+1)\tilde{k}\tilde{\varphi} + j(j+1)/2 + 1/8]} \times \\ & \times \left\{ \left[\varphi_j^k + (\varphi - \varphi_j^k) F_\alpha(\varphi)\right] d\varphi \wedge \mathbf{area}(S^2)|_{\mathbf{n}} + \mathcal{O}(t^{-1}) \right\} \end{aligned}$$

where $\tilde{\varphi} = \varphi/(2\pi)$, $F_\alpha : [0, 2\pi) \to \mathbb{R}$ is continuous (depends on α) and

$$\mathbf{area}(S^2)|_{\mathbf{n}} := n_1 dn_2 \wedge dn_3 + n_2 dn_3 \wedge dn_1 + n_3 dn_1 \wedge dn_2 = \sin\theta \, d\theta \wedge d\phi$$

is the area 2-form of S^2 at $\mathbf{n}$ (where (θ, ϕ) are the spherical coordinates).

Using the formal notation $\delta(\varphi - a)$ for the Dirac measure δ_a concentrated on $a \in (0, 2\pi)$ one gets

$$\delta(\varphi - a) = \underset{r\to\infty}{\text{w}^*\text{-lim}} \sqrt{\frac{r}{\pi}} e^{-r(\varphi-a)^2}$$

and for any $p \geq 1$,

$$\underset{r\to\infty}{\text{w}^*\text{-lim}} \frac{1}{r^p} \sqrt{r}\, e^{-r\varphi^2} = 0.$$

Since $2\tilde{k}^2\tilde{\varphi}^2 - (2j+1)\tilde{k}\tilde{\varphi} + j(j+1)/2 + 1/8 = \left(\frac{\tilde{k}}{\sqrt{2}\pi}\right)^2 (\varphi - \varphi_j^k)^2$, by putting the parameter r to $\left(\frac{\tilde{k}}{\sqrt{2}\pi}\right)^2 t$ in the above equations, we may conclude that, when $\psi^*\Theta^t[3]$ is interpreted as a measure on $[0, 2\pi] \times S^2$,

$$\underset{t\to\infty}{\text{w}^*\text{-lim}}\, \psi^*\Theta^t[3] = -\sqrt{\pi} i \left(j + \frac{1}{2}\right) \delta_{\varphi_j^k} \otimes \mathbf{area}(S^2)$$

and Theorem follows.

References

[At] M. Atiyah: K theory past and present. math.KT/0012213. Sitzungsberichte der Berliner Mathematischen Gesellschaft, 411–417, Berliner Math. Gesellschaft, Berlin, 2001.

[AS] M. Atiyah and G. Segal: Twisted K-theory. math.KT/0407054

[Bi] J.-M. Bismut: Localization formulas, superconnections, and the index theorem for families. Comm. Math. Phys. **103,** no. 1, 127–166 (1986)

[BCMMS] P. Bouwknegt, A. Carey, V. Mathai, M. K. Murray, and D. Stevenson: Twisted K-theory and K-theory of bundle gerbes. hep-th/0106194. Commun. Math. Phys. **228,** 17-49 (2002)

[Br] Volker Braun: Twisted K theory of Lie groups. hep-th/0305178. **JHEP** 0403, 029 (2004)

[CM] A. L. Carey and J. Mickelsson: The universal gerbe, Dixmier-Douady class, and gauge theory. Lett. Math. Phys. **59,** 47-60 (2002)

[CMM] A. L. Carey, J. Mickelsson, and M.K. Murray: Index theory, gerbes, and hamiltonian quantization hep-th/9511151 Commun. Math. Phys. **183,** 707-722 (1997)

[DK] P. Donovan and M. Karoubi: Graded Brauer groups and K-theory with local coefficients. Inst. Hautes tudes Sci. Publ. Math. No. **38,** 5-25 (1970)

[Do] Christopher L. Douglas: On the twisted K-homology of simple Lie groups. math.AT/0402082

[FdV] H. Freudenthal and H. de Vries: *Linear Lie Groups.* Pure Appl. Math. **35,** Academic Press, New York (1969)

[Fr] D. Freed: Twisted K-theory and loop groups. math.AT/0206237. Publ. in the proceedings of ICM2002, Beijing.

[FHT] D. Freed, M. Hopkins, and C. Teleman: Twisted equivariant K-theory with complex coefficients. math.AT/0206257. Twisted K-theory and loop group representations. math.AT/0312155

[GR] Krzysztof Gawedzki and Nuno Reis: WZW branes and gerbes. hep-th/0205233. Rev. Math. Phys. **14,** 1281-1334 (2002)

[Mi1] J. Mickelsson: Gerbes, (twisted) K theory, and the supersymmetric WZW model. hep-th/0206139. Publ. in *Infinite Dimensional Groups and Manifolds,* ed. by T. Wurzbacher. IRMA Lectures in Mathematics and Theoretical Physics **5,** Walter de Gruyter, Berlin (2004)

[Mi2] J. Mickelsson: Twisted K theory invariants. math.AT/0401130. Lett. in Math. Phys. **71,** 109-121 (2005)

[MP] J. Mickelsson and J.-P. Pellonpää: Families index theorem in supersymmetric WZW model and twisted K-theory: The $SU(2)$ case. hep-th/0509064, version 1

[PS] A. Pressley and G. Segal: *Loop Groups.* Clarendon Press, Oxford (1986)

[Qu] D. Quillen: Superconnections and the Chern character. Topology **24,** no. 1, 89–95 (1985)

[Ro] J. Rosenberg: Continuous-trace algebras from the bundle theoretic point of

view. J. Austral. Math. Soc. Ser. **A 47** no. 3, 368–381 (1989)